TRINITY COLLEGE

Mathematics Specialist Units 3 & 4
Test 6 2016

Section 1 Calculator Free

Related Rates, Incremental Formula & Solving Differential Equations.

STUDENT'S NAME: SOLUTIONS

DATE: Thursday 1st September **TIME:** 25 minutes **MARKS**: 30

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet.

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (7 marks)

(a) Determine an expression for $\frac{dy}{dt}$ given $y = e^{2x}$ and $\frac{dx}{dt} = 5$. [3]

$$y = e^{2x}$$

$$\Rightarrow \frac{dy}{dt} = \frac{d(e^{2x})}{dx} \cdot \frac{dx}{dt} \quad \checkmark$$

$$= 2e^{2x} \cdot 5 \quad \checkmark$$

$$= \underline{\underline{10e^{2x}}} \quad \checkmark$$

$$= 10y$$ (if you prefer ys to xs)

(b) If $y = \sin(2x)$ and $\frac{dx}{dt} = 3$, evaluate $\frac{dy}{dt}$ when $x = \frac{\pi}{8}$. [4]

$$y = \sin(2x)$$

$$\Rightarrow \frac{dy}{dt} = 2\cos(2x) \cdot \frac{dx}{dt} \quad \checkmark\checkmark$$

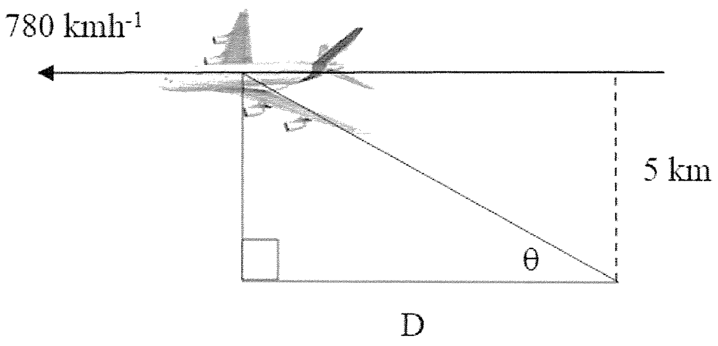
$$\Rightarrow \left.\frac{dy}{dt}\right|_{x=\frac{\pi}{8}} = 2\cos\left(\frac{\pi}{4}\right) \cdot 3 \quad \checkmark$$

$$= \frac{2}{\sqrt{2}} \cdot 3$$

$$= \underline{\underline{3\sqrt{2}}} \quad \checkmark$$

2. (10 marks)

You see a plane fly directly overhead at an altitude of 5 km. the plane is moving horizontally away from you at a constant speed of 780 kmh^{-1} with an angle of elevation of θ as shown.



(a) Show that the horizontal distance, D, between the plane and you is given by $D = \dfrac{5}{\tan(\theta)}$ [2]

$\tan\theta = \frac{5}{D}$ ✓

$\Rightarrow D\tan\theta = 5$

$\therefore D = \frac{5}{\tan\theta}$ ✓ Q.E.D.

(b) Determine the simplest expression for $\dfrac{dD}{d\theta}$. [3]

$D = 5(\tan\theta)^{-1}$

$\Rightarrow \frac{dD}{d\theta} = -5(\tan\theta)^{-2} \cdot \sec^2\theta$ ✓

$= -5\left(\frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\cos\theta}\right)^2$ ✓

$= \underline{\underline{\frac{-5}{\sin^2\theta}}}$ ✓

(c) Calculate the rate at which the angle of elevation is changing over time (in radians/hour) when $\theta = \dfrac{\pi}{6}$. [3]

$\frac{dD}{dt} = \frac{dD}{d\theta} \cdot \frac{d\theta}{dt}$ ✓

$\Rightarrow 780 = \frac{-5}{\sin^2(\frac{\pi}{6})} \cdot \frac{d\theta}{dt}$ ✓

$\Rightarrow 780 = -20 \cdot \frac{d\theta}{dt}$

$\therefore \frac{d\theta}{dt} = \underline{\underline{-39}}$ radians/hr ✓ (Decreasing)

N.B. You are expecting a negative result as the angle is decreasing.

(d) Is this a reliable measure of the rate, $\dfrac{d\theta}{dt}$, in the long run? [2]

No, ✓ because the angle will be zero after a few seconds! ✓

(N.B. This is a more urgent issue than the curvature of the earth or a drunk pilot who can't fly straight!)

3. (9 marks)

(a) Given the differential equation $\frac{1}{y^2}\frac{dy}{dx}=\frac{1}{x}$, solve for y given that when $x=e$, $y=1$.

[4]

$$\Rightarrow \int y^{-2}\,dy = \int \frac{1}{x}\,dx \quad ✓$$

$$\Rightarrow \frac{y^{-1}}{-1} = \ln|x| + c$$

$$\Rightarrow -\frac{1}{y} = \ln|x| + c \quad ✓$$

Given $y(e)=1$, $\Rightarrow -1 = \ln e + c$

$$\therefore \quad c = -2 \quad ✓$$

$$\Rightarrow -\frac{1}{y} = \ln|x| - 2$$

$$\therefore \quad y = \frac{1}{2-\ln|x|} \quad ✓$$

(b) Given that $\frac{dy}{dx}=\frac{1+y^2}{2xy}$, solve for y in terms of x, given that when $x=1$, $y=-1$.

[5]

$$\Rightarrow \int \frac{2y}{1+y^2}\,dy = \int \frac{1}{x}\,dx \quad ✓$$

$$\Rightarrow \ln(1+y^2) = \ln|x| + C_1 \quad ✓$$

$$\Rightarrow \ln(1+y^2) = \ln|x| + \ln C_2$$

ie. $C_1 = \ln C_2$

$$\Rightarrow \ln(1+y^2) = \ln|C_2 x| \quad ✓$$

log laws.

$$\Rightarrow 1+y^2 = |C_2 x| \quad ✓$$

when $x=1$, $y=-1$

$\Rightarrow C_2 = 2$

$$\Rightarrow y^2 = |2x| - 1$$

$$\Rightarrow y = \pm\sqrt{2|x|-1}$$

$$\therefore \quad y = -\sqrt{2|x|-1} \quad ✓$$

given $y(1)=-1$

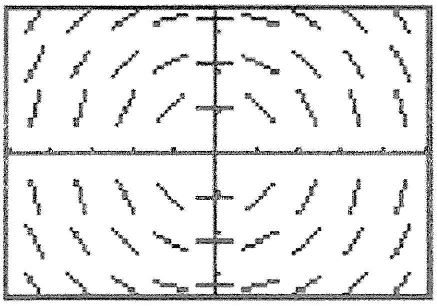
disgard $y=\sqrt{2|x|-1}$

4. (4 marks)

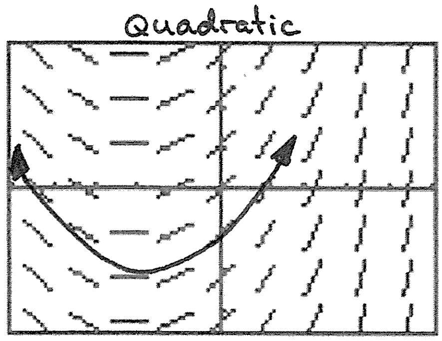
Match the slope field with the differential equation. Place the letter for the corresponding equation on the appropriate line

A. $\dfrac{dy}{dt} = \dfrac{1}{2}t + 1$ B. $\dfrac{dy}{dt} = t - y$ C. $\dfrac{dy}{dt} = y$ D. $\dfrac{dy}{dt} = -\dfrac{t}{y}$

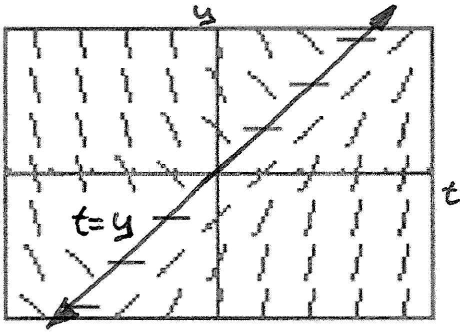
D ✓



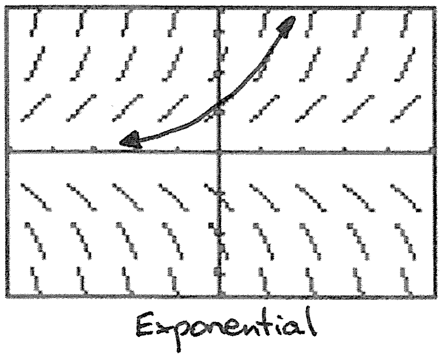
A ✓



B ✓



C ✓



In order of ease of consideration:

A: $\dfrac{dy}{dt} = \frac{1}{2}t + 1$

$\Rightarrow y = \dfrac{t^2}{4} + t + c$ Quadratic

C: $\dfrac{dy}{dt} = y$

$\Rightarrow \int \frac{1}{y}\, dy = \int dt$

$\Rightarrow \ln|y| = t + c$

$\Rightarrow y = \pm e^{t+c}$ Exponential

B: $\dfrac{dy}{dt} = t - y = 0$ when $t = y$

D: By elimination!

End of Questions

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Section 2 Calculator Assumed

Related Rates, Incremental Formula & Solving Differential Equations.

STUDENT'S NAME: SOLUTIONS

DATE: Thursday 1st September **TIME:** 25 minutes **MARKS**: 30

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet retained from Section 1.

Special Items: Drawing instruments, templates, three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment).

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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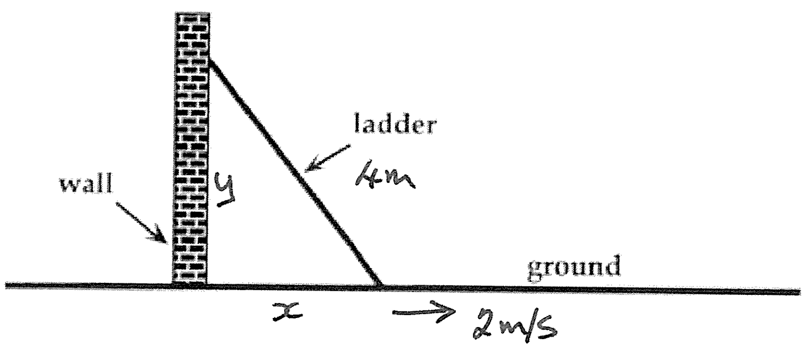
5. (6 marks)

A 4 m long ladder, standing on horizontal ground, is leaning against a vertical wall. Its base is slipping away from the wall at a constant rate of 2 m/s. At what rate, correct to 2 decimal places, will the top of the ladder be slipping down the wall when the base is 1 m out from the wall?



$x^2 + y^2 = 16$ ✓

$\Rightarrow 2x + 2y\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$ ✓

Now: $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$ ✓

$= \dfrac{-x}{y} \cdot 2$ ✓

When $x = 1$

$= \dfrac{-1}{\sqrt{15}} \cdot 2$ ✓

$= \underline{\underline{-0.52}}$ (2d.p) ✓

ie down at 0.52 m/s.

Part (e) : $100(5) \le Q < 100(20$

$500 \le Q < 2000$

6. (10 marks)

A tank contains 100 litres of brine with a concentration of 5 g/L. Fresh brine with a concentration of 20 g/L flows into the tank at a rate of 4 litres per minute. The concentration of the solution in the tank is kept uniform by constant stirring. The mixture flows out of the container at a rate of 4 litres per minute. The amount of salt at time t minutes is Q g.

Given the scenario is* modelled by the differential equation: $\frac{dQ}{dt} = 80 - \frac{Q}{25}$

A 'good' thought but this is not a logistic D.E.

(a) Show that $Q = m - ne^{-kt}$, giving the values of the constants m, n and k. [6]

Nor is this a logistic eq^n

Done for you:

$$* \quad \frac{dQ}{dt} = \text{salt Inflow} - \text{salt Outflow} = 20 \times 4 - \frac{Q}{100} \times 4 = 80 - \frac{Q}{25}$$

as given.

See Colin Rapkoch for a beautiful diagram.

$$\frac{dQ}{dt} = \frac{2000 - Q}{25}$$

$$\Rightarrow \int \frac{1}{2000-Q}\, dQ = \int \frac{1}{25}\, dt \quad ✓$$

$$\Rightarrow -\ln(2000-Q) = \frac{1}{25}t + c$$

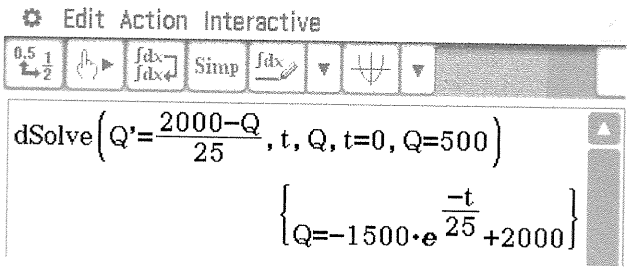
$$\Rightarrow 2000 - Q = e^{-\frac{t}{25} + c} \quad ✓$$

when $t=0$, $Q = 500 \Rightarrow 1500 = e^{c}$

$$\therefore Q = 2000 - 1500e^{-\frac{t}{25}} \quad ✓$$

ie. $m = 2000$, $n = 1500$, $k = \frac{1}{25}$ ✓ ✓ ✓

Using ClassPad dSolve to check:



(b) Determine when the concentration in the mixture in the tank reaches 6 g/L. [2]

When concentration reached 6g/L, $Q = 600$

$$600 = 2000 - 1500e^{-\frac{t}{25}} \quad ✓$$

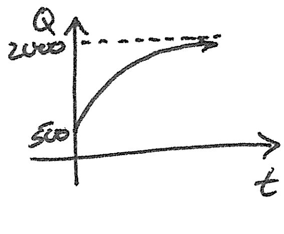
$$\Rightarrow t = 1.72 \text{ (2d.p) minutes} \quad ✓$$

(c) Determine u and v, such that for any time t, $u \le Q < v$. [2]

As $t \to \infty$, $Q \to 2000$

$\therefore 500 \le Q < 2000$

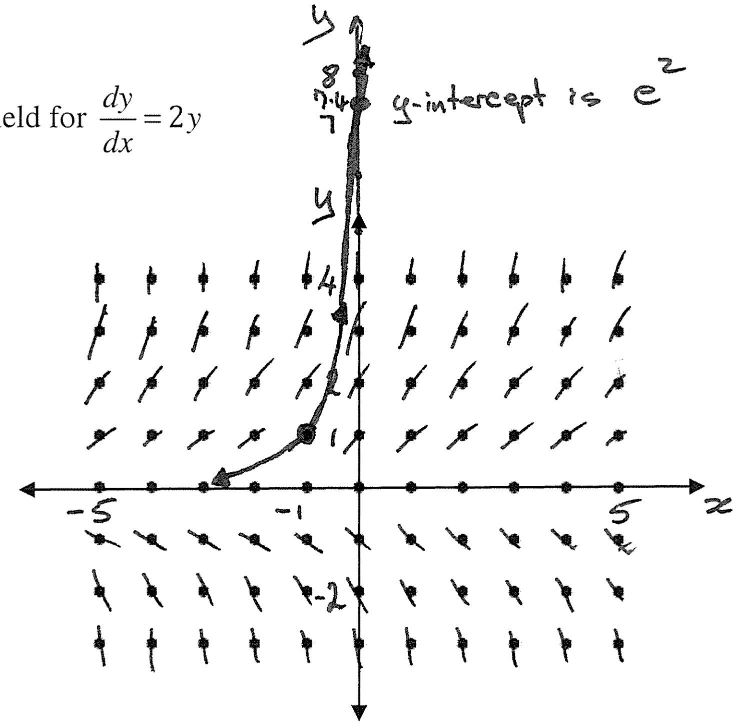
ie. $u = 500$, $v = 2000$ ✓ ✓



see alternative above

7. (6 marks)

Sketch a slope field for $\frac{dy}{dx} = 2y$ [2]

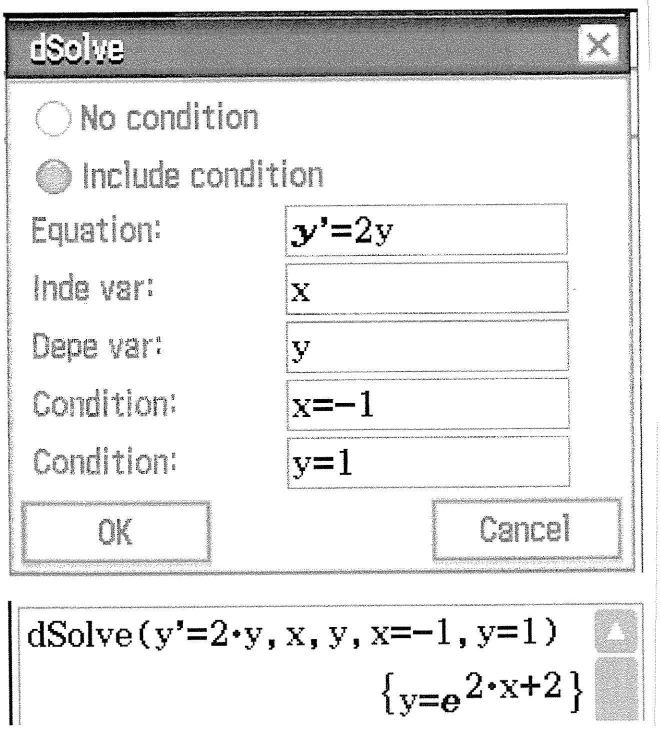


Need to show a scale.

(a) Use this slope field to sketch a solution through the point $(-1,1)$. [2]

see above ✓✓

(b) What is the particular solution to this differential equation with initial condition $y(-1)=1$? [2]



$y = e^{2x+2}$ ✓✓

i.e. $y = e^2 e^{2x}$

$\approx 7.4\, e^{2x}$

when $x=-1$

$y = e^{-2+2}$

$= e^0$

$= 1$

8. (3 marks)

The logistic differential equation, $\frac{dy}{dt} = ky\left(1 - \frac{y}{b}\right)$, has the logistic function, $y = \frac{b}{1 + Ae^{-kt}}$, as its solution.

(a) State the initial value, $y(0)$. [1]

when $t = 0$, $y = \frac{b}{1+A}$ ✓

(b) Identify the growth constant. [1]

k ✓

(c) Determine the limiting value for y, otherwise known as the carrying capacity. [1]

as $t \to \infty$, $y \to b$ ✓

9. (5 marks)

The radius of a circle increases from 20 cm to 20.1 cm.

(a) If A is the area of the circle, estimate the change in area by calculating $\frac{dA}{dr}$. [3]

$\frac{\delta A}{\delta r} \approx \frac{dA}{dr} \Rightarrow \delta A = \left.\frac{dA}{dr}\right|_{r=20} \cdot \delta r$ ✓

$A = \pi r^2$

$\Rightarrow \frac{dA}{dr} = 2\pi r$

$= 2\pi(20) \cdot 0.1$ ✓

$= \underline{\underline{4\pi}}\ cm^2$ ✓

12.566 (3d.p.)

(b) Calculate the actual area change, ΔA, and compare this with the result from part (a). [2]

$\Delta A = \pi(20.1)^2 - \pi(20)^2$

$= \pi\left((20.1)^2 - (20)^2\right)$

$= \underline{\underline{4.01\pi}}\ cm^2$ 12.598 (3d.p.)

The Incremental Formula under estimated by $0.01\pi\ cm^2$

0.0314 (3.s.f.)

End of Questions